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Structure-Preserving Stabilization for Hamiltonian System and its Applications in Solar Sail

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A structure-preserving controller is constructed to stabilize a hyperbolic Hamiltonian system. Bounded orbits for the planar solar sail three-body problem are generated by means of the controller. The invariant (stable, unstable, and center) manifolds of the equilibrium are used to stabilize a Hamiltonian system of 2 degrees of freedom using only position feedback. It is proved that the poles of the system can be assigned to any position on the imaginary axis by choosing the manifolds' gains properly. A new type of quasi-periodic orbit referred to as a stable Lissajous orbit is obtained. The orbit will degenerate to a periodic orbit in the case of resonance between modes and suitable initial values (Lyapunov orbit). Using the controller to solve the solar sail three-body problem yields a stable Lissajous orbit, which is quite different from the classical Lissajous orbit. We show that the sail equilibrium can be stabilized, and moreover that the orbit is bounded. The allocation law of the controller is also studied, which verifies that the controller is realizable.

Nomenclature

a	=	the solar pressure acceleration
G_1	=	gains of unstable manifolds
G_2	=	gains of stable manifolds
G_3	=	gains of center manifolds
$\boldsymbol{I}$	=	identical operator
$\boldsymbol{J}$	=	symplectic operator
m	=	the mass ratio between the primaries
$\boldsymbol{u}\ \bar{\boldsymbol{u}}$	=	center manifolds
$\boldsymbol{u}_\pm$	=	stable\unstable manifolds
V	=	pseudopotential function
w	=	the mean motion of the synodic (rotating) frame
α	=	attitude angle of the solar sail
β	=	the sail lightness number
$\pm\gamma i$	=	center eigenvalues
Δ	=	modification to the Coriolis acceleration
κ	=	the sensitivity of the controller
$\pm\sigma$	=	hyperbolic eigenvalues
ϕ	=	the flow generated by the Hamiltonian system
$\boldsymbol{\Xi}(T)$	=	force coefficient in the x direction

Superscript

H	=	the Hermitian transpose of a complex vector or matrix

I. Introduction

MOST classical astrodynamics problems can be classified as hyperbolic Hamiltonian systems [e.g., the circular restricted three-body problem (CR3BP), the Hill three-body problem (H3BP), and so on]. For the hyperbolic Hamiltonian system, just like the collinear libration points L_1, L_2, and L_3 in CR3BP [1], there exist hyperbolic equilibria that have stable, unstable, and center manifolds. The unstable manifolds will lead to instability in the Lyapunov sense.

Bounded motions near hyperbolic equilibria have been broadly applied to various astronautical missions [1,2], such as missions to increase the coverage rate to the object (ground station or target spacecraft) and mission that avoid communication signals being lost in the sun, such as Lissajous (or halo) orbit generated in CR3BP. Some works have also involved in the topic of bounded orbits generated by solar sail [3–7], and Morimoto et al. [8,9] developed periodic orbits using the required acceleration generated by low-thrust propulsion . However, all of the mission orbits are unstable and additional maneuvers are needed for station keeping. A simple structure-preserving controller was constructed by Scheeres [10] in which only the instantaneous stable and unstable manifolds of a system are involved, and the manifolds' gains are constrained to be equal.

In this paper, the work of Scheeres is extended. We propose a structure-preserving controller to generate a new type of quasi-periodic orbit; a stable Lissajous orbit, which is quite different from the classical Lissajous orbit for its Lyapunov stability [1,2]. A solar sail mission is an ideal application for our controller due to its zero fuel consumption. In our work not only stable and unstable manifolds are used to stabilize the system, but also center manifolds. Also, the constraints on the gains are released. Furthermore, it is proved that the poles of the system can be assigned to any different positions on the imaginary axis by the new controller.

The controller is implemented by changing the hyperbolic equilibrium (saddle) to one that is elliptic (center) with the poles on the imaginary axis (marginal stability). For a general system (non-Hamiltonian system), its poles are not expected to be assigned on the imaginary axis because the higher-order nonlinear terms may bring instability into the system. But for the Hamiltonian system, as demonstrated by means of the Morse lemma, the nonlinear controlled system is Lyapunov stable.

The stable Lissajous orbit degenerates into two types of periodic orbits in the cases of resonance and suitable initial values. The resonant orbits (type 1) appear in the case of resonance, and the Lyapunov orbits (type 2) appear for suitable initial values. A general iterative algorithm is proposed via differential correction to generate the initial values of the Lyapunov orbit. Simulation results demonstrate the validity of the structure-preserving controller.

II. Hamiltonian Dynamics

Most planar astrodynamic problems classified as Hamiltonian systems can be expressed in the following form:

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$$\begin{cases} \ddot{x} - 2\omega\dot{y} = \frac{\partial V}{\partial x} \\ \ddot{y} + 2\omega\dot{x} = \frac{\partial V}{\partial y} \end{cases} \tag{1}$$

where V is the pseudopotential function and ω is the mean motion of the synodic (rotating) frame chosen (assuming circular motion).

V and ω can be used to define the different astrodynamic problems, such as the following examples:

The circular restricted three-body problem:

$$\omega = 1, \qquad V = \frac{1}{2}(x^2 + y^2) + \frac{\mu}{r_1} + \frac{1-\mu}{r_2}$$

where μ is the mass ratio between the primaries, r_1 and r_2 are, respectively, the distances between the spacecraft and the two primaries.

The planar solar sail three-body problem:

$$\omega = 1, \qquad V = \frac{1}{2}(x^2 + y^2) + \frac{\mu}{r_1} + \frac{1-\mu}{r_2} + a \cdot (n_x \cdot x + n_y \cdot y)$$

where $\boldsymbol{n} = [n_x \quad n_y]^T$ is the sail surface normal vector.

The Hill three-body problem:

$$\omega = 1, \qquad V = \frac{3}{2}x^2 + \frac{\mu}{r}$$

The solar sail two-body problem [11]:

$$\omega = 0, \qquad V = \zeta \cdot y + \frac{\mu}{r}$$

where μ is the gravitational attraction of the central body, r is distance between the spacecraft and the body, and ζ is the maximal acceleration provided by the sail.

Equation (1) has a Lagrangian structure, thus using a Legendre transformation

$$\begin{bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ -\omega\boldsymbol{J} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \dot{\boldsymbol{r}} \end{bmatrix} \tag{2}$$

Equation (1) is transformed to the following Hamiltonian structure [10]:

$$H = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{p} + \omega\boldsymbol{p}^T\boldsymbol{J}\boldsymbol{q} + \frac{1}{2}\omega^2\boldsymbol{q}^T\boldsymbol{q} - V(\boldsymbol{q}) \tag{3}$$

where $\boldsymbol{J}$ is the symplectic operator, $\boldsymbol{I}$ is identical operator, and $\boldsymbol{r} = [x \quad y]^T$.

The equilibria of Eq. (1) can be solved from

$$\begin{cases} \dot{\boldsymbol{r}} = 0, \ddot{\boldsymbol{r}} = 0 \\ \frac{\partial V}{\partial \boldsymbol{r}} = 0 \end{cases} \tag{4}$$

Denote $V_{\boldsymbol{rr}}$ as the second derivative matrix of the pseudopotential function V to the position vector $\boldsymbol{r}$, and the element of $V_{\boldsymbol{rr}}$ has the general expression as $V_{mn} = \frac{\partial^2 V}{\partial m \partial n}$, $(m, n) = (x, y)$.

And then a hyperbolic (saddle) equilibrium is resulted from $V_{xx} \cdot V_{yy} - V_{xy}^2 < 0$ [10].

The variation equation near the equilibrium is

$$\delta\ddot{\boldsymbol{r}} - 2\omega\boldsymbol{J}\delta\dot{\boldsymbol{r}} - V_{\boldsymbol{rr}}\delta\boldsymbol{r} = 0 \tag{5a}$$

which can also be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \delta\boldsymbol{r} \\ \delta\dot{\boldsymbol{r}} \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{I} \\ V_{\boldsymbol{rr}} & 2\omega\boldsymbol{J} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{r} \\ \delta\dot{\boldsymbol{r}} \end{bmatrix} \tag{5b}$$

After the Legendre transformation, the linear Hamiltonian dynamics are obtained as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \delta\boldsymbol{q} \\ \delta\boldsymbol{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & 0 \\ -\omega\boldsymbol{J} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} 0 & \boldsymbol{I} \\ V_{\boldsymbol{rr}} & 2\omega\boldsymbol{J} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & 0 \\ \omega\boldsymbol{J} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{q} \\ \delta\boldsymbol{p} \end{bmatrix} \tag{6}$$

and the Hamiltonian is

$$\begin{aligned} H_2 &= \frac{1}{2}\begin{bmatrix} \delta\boldsymbol{q}^T & \delta\boldsymbol{p}^T \end{bmatrix} \begin{bmatrix} 0 & -\boldsymbol{I} \\ \boldsymbol{I} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & 0 \\ -\omega\boldsymbol{J} & \boldsymbol{I} \end{bmatrix} \\ &\times \begin{bmatrix} 0 & \boldsymbol{I} \\ V_{\boldsymbol{rr}} & 2\omega\boldsymbol{J} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & 0 \\ \omega\boldsymbol{J} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{q} \\ \delta\boldsymbol{p} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \delta\boldsymbol{q}^T & \delta\boldsymbol{p}^T \end{bmatrix} \begin{bmatrix} \omega^2\boldsymbol{I} - V_{\boldsymbol{rr}} & -\omega\boldsymbol{J} \\ \omega\boldsymbol{J} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{q} \\ \delta\boldsymbol{p} \end{bmatrix} \end{aligned} \tag{7}$$

III. Structure-Preserving Controller to Stabilize the Hyperbolic System

Then the characteristic equation for the linear Lagrangian dynamics is

$$\lambda^4 + B\lambda^2 + C = 0 \tag{8}$$

where

$$B = 4\omega^2 - V_{xx} - V_{yy}, \qquad C = V_{xx} \cdot V_{yy} - V_{xy}^2 \tag{9}$$

The controller will change the numerical values of V_{xx}, V_{yy}, and V_{xy} into $\tilde{V}_{xx}$, $\tilde{V}_{yy}$, and $\tilde{V}_{xy}$ (and the values of B and C are changed into $\tilde{B} = 4\omega^2 - \tilde{V}_{xx} - \tilde{V}_{yy}$ and $\tilde{C} = \tilde{V}_{xx} \cdot \tilde{V}_{yy} - \tilde{V}_{xy}^2$). Without confusion, we will still note as B and C as $\tilde{B}$ and $\tilde{C}$ for the controlled system.

For the hyperbolic system, the hyperbolic eigenvalues are $\pm\sigma$, and the stable\unstable manifolds are

$$\boldsymbol{u}_{\pm} = \frac{1}{\sqrt{1 + u_{\pm}^2}} \begin{bmatrix} 1 \\ u_{\pm} \end{bmatrix}$$

where

$$\sigma^2 = -\frac{1}{2}B + \frac{1}{2}\sqrt{B^2 - 4C}, \qquad u_{\pm} = \frac{\sigma^2 - V_{xx}}{V_{xy} \pm 2\omega\sigma} = \frac{V_{xy} \mp 2\omega\sigma}{\sigma^2 - V_{yy}}$$

the center eigenvalues are $\pm\gamma i$, and the center manifolds are $\boldsymbol{u}$ and $\bar{\boldsymbol{u}}$, with

$$\gamma^2 = \frac{1}{2}B + \frac{1}{2}\sqrt{B^2 - 4C}, \qquad \boldsymbol{u} = \frac{1}{\sqrt{1 + u\bar{u}}} \begin{bmatrix} 1 \\ u \end{bmatrix}$$

$$\bar{\boldsymbol{u}} = \frac{1}{\sqrt{1 + u\bar{u}}} \begin{bmatrix} 1 \\ \bar{u} \end{bmatrix}, \qquad u = \frac{-\gamma^2 - V_{xx}}{V_{xy} + 2\omega\gamma i} = \frac{V_{xy} - 2\omega\gamma i}{-\gamma^2 - V_{yy}}$$

So the eigenvalues and their manifolds have the relationships as follows:

$$[\sigma^2\boldsymbol{I} \mp 2\omega\sigma\boldsymbol{J} - V_{\boldsymbol{rr}}]\boldsymbol{u}_{\pm} = 0 \tag{10a}$$

$$\begin{cases} [-\gamma^2\boldsymbol{I} + 2\omega\gamma i\boldsymbol{J} - V_{\boldsymbol{rr}}]\boldsymbol{u} = 0 \\ [-\gamma^2\boldsymbol{I} - 2\omega\gamma i\boldsymbol{J} - V_{\boldsymbol{rr}}]\bar{\boldsymbol{u}} = 0 \end{cases} \tag{10b}$$

Then the controller is constructed as

$$\begin{aligned} \boldsymbol{T}_C = \Big\{ &-\sigma^2\Big[G_1\boldsymbol{u}_+\boldsymbol{u}_+^T + G_2\boldsymbol{u}_-\boldsymbol{u}_-^T\Big] \\ &- \gamma^2 G_3[\boldsymbol{u}\boldsymbol{u}^H + \bar{\boldsymbol{u}}\bar{\boldsymbol{u}}^H]\Big\}\delta\boldsymbol{r} + 2\Delta \cdot \boldsymbol{J}\delta\dot{\boldsymbol{r}} \end{aligned} \tag{11}$$

Compared with Scheeres' controller [10], the controller has some important improvements:

1) The position feedback can stabilize the system: the stable and unstable manifolds used to stabilize the system can have different gains G_1 and G_2 (see Proposition 1); the center manifolds can be used to stabilize the system (see Proposition 2).

2) Some combinations of the gains (G_3, $G_1 + G_2$, $G_1 + G_3$, $G_2 + G_3$, and $G_1 + G_2 + G_3$) can stabilize the system, but only single G_1 or G_2 cannot stabilize the system.

3) The poles can be assigned at any position on the imaginary axis, and the G_1, G_2, and G_3 required to stabilize the system are not unique (see Proposition 3).

4) The Coriolis acceleration can be modified by $\tilde{\omega} = \omega + \Delta$, which cannot stabilize the system independently because $2\Delta \cdot \boldsymbol{J}\delta\dot{\boldsymbol{r}}$ does not change V_{xx}, V_{yy}, and V_{xy}.

From 1 to 3, we have obtained that the controller will work well just via the estimation of relative position, and the estimation of relative velocity is not necessary, which is too attractive for the controller.

Proposition 1: The stable and unstable manifolds can be used to stabilize the system.

Proof: the conditions of linear stability for this system are

$$B > 0 \tag{12a}$$

$$C > 0 \tag{12b}$$

$$B^2 - 4C > 0 \tag{12c}$$

We will present why $B^2 - 4C = 0$ can bring instability into the system later.

V_{xx}, V_{yy} and V_{xy} are changed by the controller as

$$\tilde{V}_{xx} = V_{xx} - \sigma^2\left[\frac{G_1}{1+u_+^2} + \frac{G_2}{1+u_-^2}\right] \tag{13a}$$

$$\tilde{V}_{yy} = V_{yy} - \sigma^2\left[\frac{G_1 u_+^2}{1+u_+^2} + \frac{G_2 u_-^2}{1+u_-^2}\right] \tag{13b}$$

$$\tilde{V}_{xy} = V_{xy} - \sigma^2\left[\frac{G_1 u_+}{1+u_+^2} + \frac{G_2 u_-}{1+u_-^2}\right] \tag{13c}$$

For large enough G_1 and G_2, it is easy to prove the conditions in Eqs. (12a) and (12c) [10]. Next, expanding the condition in Eq. (12b) yields

$$\begin{aligned} C = {} & \sigma^4 G_1 G_2 (u_+ - u_-)^2 + \sigma^2\Big[2V_{xy}\Big\{\Big(1+u_-^2\Big)u_+ G_1 \\ & + \Big(1+u_+^2\Big)u_- G_2\Big\} - V_{xx}\Big\{\Big(1+u_-^2\Big)u_+^2 G_1 \\ & + \Big(1+u_+^2\Big)u_-^2 G_2\Big\} - V_{yy}\Big\{\Big(1+u_-^2\Big)G_1 + \Big(1+u_+^2\Big)G_2\Big\}\Big] \\ & + \Big(1+u_-^2\Big)\Big(1+u_-^2\Big)\Big(V_{xx}\cdot V_{yy} - V_{xy}^2\Big) > 0 \end{aligned} \tag{14}$$

u_+ and u_- are different because they have different eigenvalues $+\sigma$ and $-\sigma$. So the fact that $G_1 G_2 (u_+ - u_-)^2 > 0$ and the product of G_1 and G_2 indicate that G_1 and G_2 can be chosen large enough to guarantee the stability. The null for any one of G_1 and G_2 will lead to the failure of stabilization.

Proposition 2: The center manifolds can be used to stabilize the system.

Proof: Just like the proof of Proposition 1, we begin from the condition in Eq. (12b).

V_{xx}, V_{yy}, and V_{xy} are changed by the controller as

$$\tilde{V}_{xx} = V_{xx} - \gamma^2 G_3 \frac{2}{1+u\bar{u}} \tag{15a}$$

$$\tilde{V}_{yy} = V_{yy} - \gamma^2 G_3 \frac{2u\bar{u}}{1+u\bar{u}} \tag{15b}$$

$$\tilde{V}_{xy} = V_{xy} - \gamma^2 G_3 \frac{u+\bar{u}}{1+u\bar{u}} \tag{15c}$$

Expanding the condition in Eq. (12b) yields

$$\begin{aligned} C = {} & -\gamma^4 G_3^2 (u-\bar{u})^2 + 2\gamma^2 G_3 [V_{xy}(1+u\bar{u})(u+\bar{u}) \\ & - V_{xx}(1+u\bar{u})u\bar{u} - V_{yy}(1+u\bar{u})] \\ & + (1+u\bar{u})^2\Big(V_{xx}\cdot V_{yy} - V_{xy}^2\Big) > 0 \end{aligned} \tag{16}$$

Note that because $(u-\bar{u})^2 < 0$ for imaginary numbers, we can choose G_3 large enough to guarantee the stability.

Proposition 3 (theorem of poles assignment): The two poles can be assigned at any different positions on the imaginary axis, and the G_1, G_2, and G_3 required to stabilize the system are not unique.

Proof: First, we will prove the first half of the proposition.

Let

$$\boldsymbol{A} = \begin{bmatrix} \frac{\sigma^2}{1+u_+^2} & \frac{\sigma^2}{1+u_-^2} & \frac{2\gamma^2}{1+u\bar{u}} \\ \frac{\sigma^2 u_+^2}{1+u_+^2} & \frac{\sigma^2 u_-^2}{1+u_-^2} & \frac{2\gamma^2 u\bar{u}}{1+u\bar{u}} \\ \frac{\sigma^2 u_+}{1+u_+^2} & \frac{\sigma^2 u_-}{1+u_-^2} & \frac{\gamma^2(u+\bar{u})}{1+u\bar{u}} \end{bmatrix} \tag{17}$$

then V_{xx}, V_{yy}, and V_{xy} are changed by the controller as

$$\begin{bmatrix} \tilde{V}_{xx} \\ \tilde{V}_{yy} \\ \tilde{V}_{xy} \end{bmatrix} = \begin{bmatrix} V_{xx} \\ V_{yy} \\ V_{xy} \end{bmatrix} - \boldsymbol{A}\begin{bmatrix} G_1 \\ G_2 \\ G_3 \end{bmatrix} \tag{18}$$

$\tilde{V}_{xx}$, $\tilde{V}_{yy}$, and $\tilde{V}_{xy}$ can be assigned arbitrarily to satisfy the linearly stable conditions, as long as $\boldsymbol{A}$ is not regular. Obviously, $\boldsymbol{A}$ cannot be regular due to the fact that u_+ and u_- are different for the different eigenvalues $+\sigma$ and $-\sigma$.

Denote the solutions to Eq. (8) as λ_1^2 and λ_2^2, and then B and C can be determined as

$$\begin{cases} B = -\Big(\lambda_1^2 + \lambda_2^2\Big) \\ C = \lambda_1^2 \cdot \lambda_2^2 \end{cases} \tag{19}$$

then $\tilde{V}_{xx}$, $\tilde{V}_{yy}$, and $\tilde{V}_{xy}$ can be solved from the Eq. (9).

The range for λ_1^2 and λ_2^2 is

$$0 > \lambda_1^2 \geq -\frac{1}{2}B \geq \lambda_2^2 \tag{20}$$

but if one sets

$$\lambda_1^2 = \lambda_2^2 = -\frac{1}{2}B \tag{21}$$

then $(\lambda + i\sqrt{B/2})^2(\lambda - i\sqrt{B/2})^2 = 0$. The system has elementary factors with second order, and so the matrix

$$\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ V_{rr} & 2\omega\boldsymbol{J} \end{bmatrix}$$

can not be diagonalized, and the Jordan form will bring long-term dispersions.

Next, we will prove the latter half of the proposition.

The nonuniqueness of G_1, G_2, and G_3 are due to the fact that $\tilde{V}_{xx}$, $\tilde{V}_{yy}$, and $\tilde{V}_{xy}$ solved from the Eq. (9) are not unique.

Assume that both of the two sets $\tilde{V}_{xx}$, $\tilde{V}_{yy}$, $\tilde{V}_{xy}$ and $\bar{V}_{xx}$, $\bar{V}_{yy}$, $\bar{V}_{xy}$ *generate* the same B and C in Eq. (9), and the two sets have the following relationships as

$$\begin{cases} \bar{V}_{xx} + \bar{V}_{yy} = \tilde{V}_{xx} + \tilde{V}_{yy} \\ \bar{V}_{xx}\cdot\bar{V}_{yy} - \bar{V}_{xy}^2 = \tilde{V}_{xx}\cdot\tilde{V}_{yy} - \tilde{V}_{xy}^2 \end{cases} \tag{22}$$

We can obtain

$$\bar{V}_{xx}^2 - (\tilde{V}_{xx} + \tilde{V}_{yy})\bar{V}_{xx} + \Big(\bar{V}_{xy}^2 + \tilde{V}_{xx}\tilde{V}_{yy} - \tilde{V}_{xy}^2\Big) = 0 \tag{23}$$

and the real solution of the equation is

$$(\tilde{V}_{xx} - \tilde{V}_{yy})^2 + 4\left(\tilde{V}_{xy}^2 - \bar{V}_{xy}^2\right) \geq 0 \tag{24}$$

and so one can choose $\tilde{V}_{xy}^2 < \bar{V}_{xy}^2$ to get two different sets generated from Eq. (9). Further, the fact that there exist different sets that generate the same B and C indicates the nonuniqueness of G_1, G_2, and G_3.

Remark 1: If the relative velocity feedback is introduced into the controller, $\tilde{V}_{xx}$, $\tilde{V}_{yy}$, $\tilde{V}_{xy}$, and $\tilde{\omega}$ can also be assigned arbitrarily, and G_1, G_2, G_3, and Δ required by the controller are not unique either.

Remark 2: The geometric interpretation of the controller is as follows: the invariant manifolds are used to stabilize the system; G_1 and G_2 will feedback the unstable and stable manifolds to remove the unstable component of motion; and G_3 will feedback the center manifolds to augment the oscillation of motion. If G_3 is large enough, the oscillation is augmented so largely that only G_3 can stabilize the system; G_1 and G_2 cannot stabilize the system independently for the volume-preserving flow of a Hamiltonian system. However, the combination $G_1 + G_3$ (or $G_2 + G_3$) can stabilize the system because the feedback of center manifolds will compensate the absence of stable (or unstable) manifolds.

It should be emphasized that the controller developed here can be extended to the 3-DOF (degree-of-freedom) Hamiltonian system. For this system, there exist one-dimensional stable manifolds, one-dimensional unstable manifolds, and four-dimensional center manifolds, and so the manifolds' gains for feedback can be classified as G_1, G_2, G_3, and G_4. Numerical implementation of this has indicated that no simple $G_i (i = 1, 2, 3, 4)$ can stabilize the system; only the double-combination as $G_1 + G_2$, $G_3 + G_4$, $G_1 + G_3$, $G_2 + G_3$, $G_1 + G_4$, and $G_2 + G_4$ can stabilize the system; also any triplicate combination or quadruple combination can stabilize the system. However, the rigorous proof for this problem has not yet been presented so far and is still an open problem.

IV. Stability for Nonlinear Controlled Dynamics System

Let

$$\boldsymbol{T} = -\sigma^2 \left\lfloor G_1 \boldsymbol{u}_+ \boldsymbol{u}_+^T + G_2 \boldsymbol{u}_- \boldsymbol{u}_-^T \right\rfloor - \gamma^2 G_3 \lfloor \boldsymbol{u}\boldsymbol{u}^H + \bar{\boldsymbol{u}}\bar{\boldsymbol{u}}^H \rfloor \tag{25}$$

$$\boldsymbol{K} = 2\Delta \boldsymbol{J} \tag{26}$$

then the controller has the form

$$\boldsymbol{T}_C = \boldsymbol{T}\delta\boldsymbol{r} + \boldsymbol{K}\delta\dot{\boldsymbol{r}} \tag{27}$$

The fact that $\boldsymbol{T}$ is a symmetry matrix and $\boldsymbol{K}$ is a skew symmetry matrix guarantees the linear feedback controller to preserve the Hamiltonian structure [12].

Then the controlled Hamiltonian is modified as

$$H = \frac{1}{2}\boldsymbol{p}^T\boldsymbol{p} + \tilde{\omega}\boldsymbol{p}^T\boldsymbol{J}\boldsymbol{q} + \frac{1}{2}\tilde{\omega}^2\boldsymbol{q}^T\boldsymbol{q} - V(\boldsymbol{q}) - \frac{1}{2}\delta\boldsymbol{q}^T\boldsymbol{T}\delta\boldsymbol{q} \tag{28}$$

Expanding H near the equilibrium yields

$$H = H_0 + \frac{1}{2}\begin{bmatrix} \delta\boldsymbol{q}^T & \delta\boldsymbol{p}^T \end{bmatrix} \begin{bmatrix} \omega^2\boldsymbol{I} - V_{rr} & -\tilde{\omega}\boldsymbol{J} \\ \tilde{\omega}\boldsymbol{J} & \boldsymbol{I} \end{bmatrix} \times \begin{bmatrix} \delta\boldsymbol{q} \\ \delta\boldsymbol{p} \end{bmatrix} - \frac{1}{2}\delta\boldsymbol{q}^T\boldsymbol{T}\delta\boldsymbol{q} + O(3) \tag{29}$$

where H_0 is the Hamiltonian value of the equilibrium and the first order polynomial disappears due to the equilibrium.

According to Propositions 1, 2 and 3, the controller developed here can transform the hyperbolic equilibrium (saddle) to an elliptic one (center). For the elliptic equilibrium, there exists a linear symplectic transformation that transforms H to the following form:

$$\tilde{H}(\tilde{\boldsymbol{q}}, \tilde{\boldsymbol{p}}) = H_0 + \frac{1}{2}\lambda_1\left(\tilde{q}_1^2 + \tilde{p}_1^2\right) + \frac{1}{2}\lambda_2\left(\tilde{q}_2^2 + \tilde{p}_2^2\right) + O(3) \tag{30}$$

By the Morse lemma, in the neighborhood of the equilibrium there exists an analytic diffeomorphism [13], which transforms $\tilde{H}$ to the form

$$K(\bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}) = K(\bar{\boldsymbol{q}}_0, \bar{\boldsymbol{p}}_0) + \bar{\boldsymbol{q}}^T\bar{\boldsymbol{q}} + \bar{\boldsymbol{p}}^T\bar{\boldsymbol{p}} \tag{31}$$

hence the energy surfaces are locally diffeomorphic to a family of spheres that shrink down to the equilibria as $K \to K(\bar{\boldsymbol{q}}_0, \bar{\boldsymbol{p}}_0)$. Because the trajectories are tangent to the energy surfaces, the Lyapunov stability follows [13]. Thus, we have obtained the stability for nonlinear full dynamics.

Generally, the trajectories generated by the controller preserving the Hamiltonian structure are quasi periodic and Lyapunov stable, which are referred to as stable Lissajous orbits in this paper.

V. Resonance and Periodic Orbits

The Lissajous orbit, at some appropriate conditions, will degenerate into periodic orbit, which classified as resonant periodic orbit (type 1) and Lyapunov orbit (type 2).

A. Resonant Periodic Orbit

When the eigenvalues of a controlled system have the relationship of $\lambda_1{:}\lambda_2 = m{:}n$ ($m < n$, m, and n are reduction), a resonance occurs. In this case, the trajectories have some periodicity for any initial values of the spacecraft's orbit. Next, we will give a particular solution for resonance.

For $\lambda_1{:}\lambda_2 = m{:}n$, then

$$m^2(B + \sqrt{B^2 - 4C}) = n^2(B - \sqrt{B^2 - 4C}) \tag{32}$$

thus

$$mn(4\tilde{\omega}^2 - \tilde{V}_{xx} - \tilde{V}_{yy}) = (m^2 + n^2)\sqrt{\tilde{V}_{xx}\tilde{V}_{yy} - \tilde{V}_{xy}^2} \tag{33}$$

A particular solution can be constructed as

$$\begin{cases} \tilde{V}_{xx} = \tilde{V}_{yy} = \tilde{\omega}^2 \\ \tilde{V}_{xy} = \pm\tilde{\omega}^2 \cdot \frac{n^2 - m^2}{m^2 + n^2} \end{cases} \tag{34}$$

If $\tilde{\omega} = 0$, a particular solution can be constructed as

$$\begin{cases} \tilde{V}_{xx} = \tilde{V}_{yy} = -\hat{\omega}^2 \\ \tilde{V}_{xy} = \pm\hat{\omega}^2 \cdot \frac{n^2 - m^2}{m^2 + n^2} \end{cases} \tag{35}$$

where $\hat{\omega}$ is an arbitrary parameter.

Remark 3: Strictly speaking, the resonant orbit is just near periodic. An excursion from periodicity is generated during every period, and the larger the orbital amplitude is, the larger the excursion is.

B. Nonresonant Periodic Orbit

When $\lambda_1{:}\lambda_2$ is an irrational number, the type of periodic orbit referred to as a Lyapunov orbit will emerge at some special initial values of the spacecraft's orbit, which is guaranteed by the Lyapunov Center Theorem [13].

The general iterative algorithm for generating Lyapunov orbits is proposed via differential correction as follows:

Let $\boldsymbol{X} = [x \quad y \quad \dot{x} \quad \dot{y}]^T$, and ϕ is the flow generated by the system. The motion is expected to terminate at the Poincaré section $\boldsymbol{X} = [x_0 \quad 0 \quad 0 \quad 0]^T$, and the integrating time is not fixed (e.g., T for $\boldsymbol{X}_0$ and $T + \delta T$ for $\boldsymbol{X}_0 + \delta\boldsymbol{X}_0$). So the difference between the initial and final values is

$$\phi_T(\boldsymbol{X}_0) - \phi_0(\boldsymbol{X}_0) = [0 \quad \Delta y \quad \Delta\dot{x} \quad \Delta\dot{y}] = \boldsymbol{X}_d \tag{36}$$

Generally, $\boldsymbol{X}_d \neq 0$ for $\boldsymbol{X}_0$ is not exactly on the Lyapunov orbit.

Supposing that the correction to $\boldsymbol{X}_0$ is $\delta\boldsymbol{X}_0$, which achieves $\boldsymbol{X}_d = 0$, so that one can get

$$\phi_{T+\delta T}(\boldsymbol{X}_0 + \delta\boldsymbol{X}_0) = \phi_0(\boldsymbol{X}_0 + \delta\boldsymbol{X}_0) = \boldsymbol{X}_0 + \delta\boldsymbol{X}_0 \tag{37}$$

substituting Eq. (36) into Eq. (37) yields

$$\phi_{T+\delta T}(\boldsymbol{X}_0 + \delta\boldsymbol{X}_0) - \phi_T(\boldsymbol{X}_0) + \boldsymbol{X}_0 = \boldsymbol{X}_0 + \delta\boldsymbol{X}_0 - \boldsymbol{X}_d \tag{38}$$

expanding the left hand of Eq. (38) yields

$$\phi_{T+\delta T}(\boldsymbol{X}_0 + \delta\boldsymbol{X}_0) - \phi_T(\boldsymbol{X}_0) = \frac{\partial\phi_T}{\partial\boldsymbol{X}_0}\delta\boldsymbol{X}_0 + \frac{\partial\phi_T}{\partial t}\delta T \tag{39}$$

where $\frac{\partial\phi_T}{\partial t} = [\dot{x}_T \quad \dot{y}_T \quad \ddot{x}_T \quad \ddot{y}_T]^T$.

The monodromy matrix for ϕ is $\Xi(T) = \frac{\partial\phi_T}{\partial X_0}$, which can be solved from

$$\begin{cases} \Xi(t) = \frac{\partial^2\phi_t}{\partial X^2}\dot{\Xi}(t) \\ \Xi(0) = \boldsymbol{I}_{4\times4} \end{cases} \tag{40}$$

Let

$$\boldsymbol{\Phi} = \frac{\partial\phi_T}{\partial\boldsymbol{X}_0} - \boldsymbol{I} \tag{41}$$

$$\boldsymbol{D}_m = \begin{bmatrix} \boldsymbol{\Phi}_{12} & \boldsymbol{\Phi}_{13} & \boldsymbol{\Phi}_{14} & \dot{x}_T \\ \boldsymbol{\Phi}_{22} & \boldsymbol{\Phi}_{23} & \boldsymbol{\Phi}_{24} & \dot{y}_T \\ \boldsymbol{\Phi}_{32} & \boldsymbol{\Phi}_{33} & \boldsymbol{\Phi}_{34} & \ddot{x}_T \\ \boldsymbol{\Phi}_{42} & \boldsymbol{\Phi}_{43} & \boldsymbol{\Phi}_{44} & \ddot{y}_T \end{bmatrix} \tag{42}$$

where $\boldsymbol{\Phi}_{ij}$ is the component of $\boldsymbol{\Phi}$ in the ith row and jth column.

Then the correction to $\boldsymbol{X}_0$ can be formulated as

$$[\delta y_0 \quad \delta\dot{x}_0 \quad \delta\dot{y}_0 \quad \delta T]^T = \boldsymbol{D}_m^{-1}\cdot(-\boldsymbol{X}_d) \tag{43}$$

The first-order expansion in Eq. (39) is used to solve $\delta\boldsymbol{X}_0$, and so the accurate initial values should be achieved by the procedure developed previously. The numerical implementation indicates that 4–5 iterations are required.

The iterative algorithm is convergent when the initial value of the first iteration is a good approximation of the truth value, and the value from the linear oscillations generated by the Eq. (29) or Eq. (30) can be used as the initial guess values of the iteration.

VI. Some Techniques for Optimization

Proposition 3 show that G_1, G_2, G_3, and Δ required to stabilize the system are not unique, and so the rule for choosing them needs to be investigated.

Denote $\bar{U}_{\lambda_1,\lambda_2}$ as the collection of G_1, G_2, G_3, and Δ, which assign the controlled system with the expected poles λ_1 and λ_2.

Then the output of controller can be expressed as

$$\boldsymbol{T}_C = \boldsymbol{T}\cdot\delta\boldsymbol{r} + \boldsymbol{K}\cdot\delta\dot{\boldsymbol{r}} = \begin{bmatrix} \boldsymbol{T} & 0 \\ 0 & \boldsymbol{K} \end{bmatrix}\begin{bmatrix} \delta\boldsymbol{r} \\ \delta\dot{\boldsymbol{r}} \end{bmatrix} \tag{44}$$

Let $z = [\delta\boldsymbol{r}^T \quad \delta\dot{\boldsymbol{r}}^T]^T$, then the sensitivity of the controller is defined as $\kappa = \frac{\|\boldsymbol{T}_C\|_2}{\|z\|_2}$.

The Frobenius norm is consistent to the Euclidean norm of the vector and can be used to measure the sensitivity of the controller as

$$\kappa = \frac{\|\boldsymbol{T}_C\|_2}{\|z\|_2} \le \left\| \begin{bmatrix} \boldsymbol{T} & 0 \\ 0 & \boldsymbol{K} \end{bmatrix} \right\|_F = \sqrt{\mathrm{tr}(\boldsymbol{T}^T\boldsymbol{T}) + 8\Delta^2} \tag{45}$$

Let

$$\boldsymbol{G}_m = \begin{bmatrix} G_1 & G_2 & G_3 & G_1 & G_2 & G_3 \\ G_1 & G_2 & G_3 & G_1 & G_2 & G_3 \end{bmatrix}^T \tag{46}$$

$$\boldsymbol{A}_m = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \boldsymbol{A}_{13} & \boldsymbol{A}_{31} & \boldsymbol{A}_{32} & \boldsymbol{A}_{33} \\ \boldsymbol{A}_{31} & \boldsymbol{A}_{32} & \boldsymbol{A}_{33} & \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \boldsymbol{A}_{23} \end{bmatrix} \tag{47}$$

Decompose $\boldsymbol{T}$ as

$$\boldsymbol{T} = \begin{bmatrix} G_1\cdot\frac{\sigma^2}{1+u_+^2} + G_2\cdot\frac{\sigma^2}{1+u_-^2} + G_3\cdot\frac{2\gamma^2}{1+u\bar{u}} & G_1\cdot\frac{\sigma^2 u_+}{1+u_+^2} + G_2\cdot\frac{\sigma^2 u_-}{1+u_-^2} + G_3\cdot\frac{\gamma^2(u+\bar{u})}{1+u\bar{u}} \\ G_1\cdot\frac{\sigma^2 u_+}{1+u_+^2} + G_2\cdot\frac{\sigma^2 u_-}{1+u_-^2} + G_3\cdot\frac{\gamma^2(u+\bar{u})}{1+u\bar{u}} & G_1\cdot\frac{\sigma^2 u_+^2}{1+u_+^2} + G_2\cdot\frac{\sigma^2 u_-^2}{1+u_-^2} + G_3\cdot\frac{2\gamma^2 u\bar{u}}{1+u\bar{u}} \end{bmatrix} = \boldsymbol{A}_m\cdot\boldsymbol{G}_m \tag{48}$$

thus

$$\begin{aligned} \|\mathbf{T}\|_F^2 &= [\mathrm{tr}(\mathbf{T}^T\mathbf{T})]^2 \le \|\mathbf{A}_m\|_F^2\cdot\|\mathbf{G}_m\|_F^2 \\ &= \|\mathbf{A}_m\|_F^2\cdot 4(G_1^2 + G_2^2 + G_3^2) \end{aligned} \tag{49}$$

and substituting Eq (49) into Eq. (45) yields

$$\kappa \le \sqrt{\|\mathbf{A}_m\|_F^2\cdot 4\left(G_1^2 + G_2^2 + G_3^2\right) + 8\Delta^2} \tag{50}$$

$\|\boldsymbol{A}_m\|_F$ can be calculated directly from the constant matrix $\boldsymbol{A}_m$ involving the invariant manifolds of the equilibrium.

For full feedback, $L = \|\boldsymbol{A}_m\|_F^2\cdot 4(G_1^2 + G_2^2 + G_3^2) + 8\Delta^2$ can be chosen as the optimization index to choose more suitable values for the manifolds' gains; in the case of only the position feedback, the optimization index can be chosen as $L = G_1^2 + G_2^2 + G_3^2$.

The problem is classified as nonlinear constrained optimization with the constraint Eq. (19), namely

$$\left.\begin{array}{c} \min\limits_{(G_1,G_2,G_3,\Delta)\in\bar{U}_{\lambda_1,\lambda_2}} \\ \text{s.t. } B + \left(\lambda_1^2 + \lambda_2^2\right) = 0 \\ L, C - \lambda_1^2\cdot\lambda_2^2 = 0 \end{array}\right\}$$

The functions "fmincon" and "confuneq" in MATLAB® can deal with the optimization.

Remark 4: the hyperbolic equilibrium has one-dimensional stable, one-dimensional unstable, and two-dimensional center manifolds,

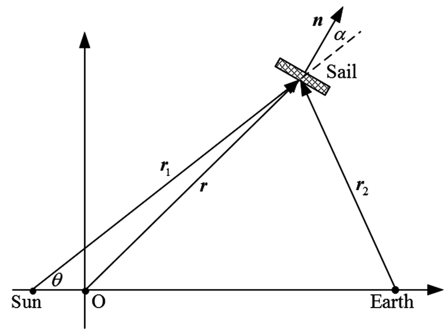

Fig. 1 Schematic geometry of the sun–Earth/moon–sail three-body system.

but the weight for manifolds' gains are not accordant with their dimensions, namely, L does not have the form $L = G_1^2 + G_2^2 + 2G_3^2$; for full feedback, the weight of Δ is much smaller than G_1, G_2 and G_3.

VII. Application to the Solar Sail Three-Body Problem

A solar sail is a spacecraft without fuel consumption, and its orbital controls come from the solar radiation pressure by orientating the attitude relative to the sun. Thus the maintaining control for the solar sail can neglect the control consumption and implement the complicated control law, which is incomparable to the impulse propulsion or low-thrust ion propulsion.

A. Sail Equilibria

The sail, associated with the sun and the Earth–moon system (which is regarded as one whole celestial body, denoted as Earth/moon), is referred to as a solar sail restricted three-body system. Here we assume the sun–Earth/moon system revolves in a circular motion, as shown in Fig. 1.

The dynamics in a synodic (rotating) frame are

$$\ddot{\boldsymbol{r}} - 2\boldsymbol{J}\dot{\boldsymbol{r}} - \nabla U = \boldsymbol{a} \tag{51}$$

and the solar pressure acceleration is

$$\boldsymbol{a} = \beta \frac{1-\mu}{r_1^4} (\boldsymbol{r}_1 \cdot \boldsymbol{n})^2 \boldsymbol{n} \tag{52}$$

where $U(\boldsymbol{r}) = \frac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{1-\mu}{r_2}$; r_1 and r_2 are, respectively, the distances between the spacecraft and the sun and Earth/moon; β is the sail lightness number, which is the ratio of the radiation pressure acceleration to gravitational acceleration.

Obviously, equilibria exist in this coordinate system. There are five equilibria (referenced as Lagrange points) in classical restricted three-body problem, however, infinite equilibria exist in the solar sail restricted three-body system.

The sail equilibria can be determined as [3,4]

$$-\nabla U = \boldsymbol{a} \tag{53a}$$

$$\boldsymbol{n} = \frac{-\nabla U}{\| -\nabla U \|} \tag{53b}$$

$$\beta = \frac{r_1^4}{1-\mu} \cdot \frac{\| -\nabla U \|^3}{(\boldsymbol{r}_1 \cdot -\nabla U)^2} \tag{53c}$$

A set of sail equilibria can be parameterized by the sail lightness number β. This parameterization for sun–Earth/moon system generates level surfaces as demonstrated, in Fig. 2.

The black dot in the figure is the equilibrium used to design the stable Lissajous orbit, in this section $z_0 = [0.75 \quad 0.25 \quad 0 \quad 0]^T$ ($\beta_0 = 0.5059$).

B. Stable Lissajous Orbit

There are quasi-periodic or periodic orbits near the sail equilibria. Some types of periodic orbits have been proposed in the solar sail restricted three-body problem. McInnes [3] first applied the classic theory of libration point orbit to generate the halo orbit near the on-axis sail equilibrium. Then Baoyin [5] obtained two different types of halo orbits near the on-axis sail equilibrium, just along the ideas of McInnes, Bookless [7] developed a control strategy to keep the Lissajous orbit generated from the linear dynamics of sail. For the off-axis sail equilibrium, McInnes [3] tried to generate Lissajous orbits near it but failed in maintaining a bounded trajectory for a long time. Waters [6] applied the classic theory to generate some halo orbits, which are only available for the off-axis sail equilibria in some specified regions.

Furthermore, all the interesting orbits generated by researchers are unstable and have one-dimensional unstable manifolds, so that the station keeping strategies are necessary.

Now we will investigate the dynamics of sail and modify the pseudopotential function as

$$V = \frac{1}{2}(x^2 + y^2) + \frac{\mu_1}{r_1} + \frac{1-\mu}{r_2} + \boldsymbol{a} \cdot \boldsymbol{r} \tag{54}$$

where $\boldsymbol{a}$ is the solar pressure acceleration required by the equilibrium. Obviously, the expected equilibrium is the hyperbolic equilibrium for the system generated by Eq. (54). The modification for the pseudopotential function is quite different from the McInnes modification [3], which leads to his failure. To generate the bounded orbit, McInnes assigned the poles of controlled system on the

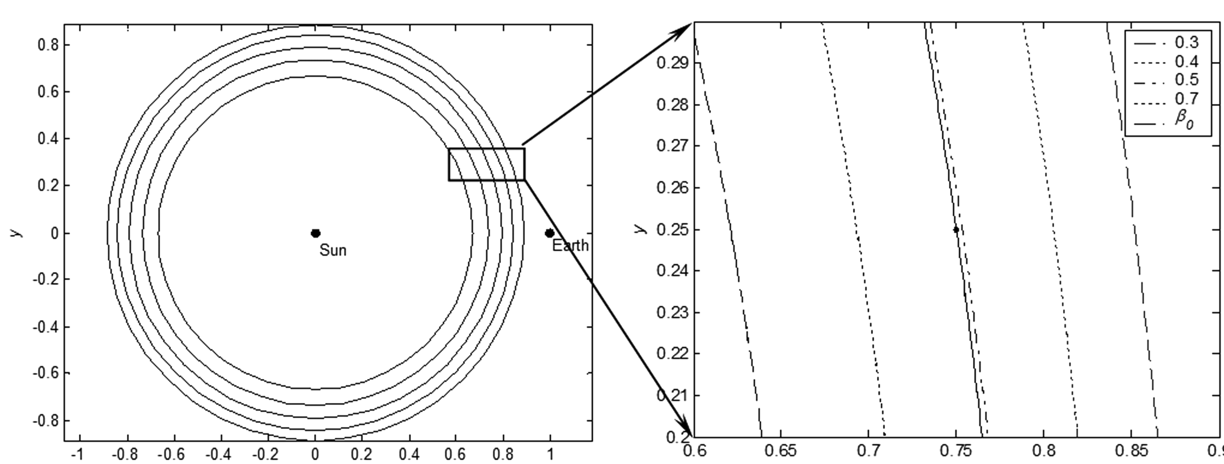

Fig. 2 Section of the level surfaces in sun–Earth/moon system.

imaginary axis, but his controller did not hold the Hamiltonian structure; but for the dissipative system, McInnes's assignment is unstable.

Furthermore, quite different from Waters [6], the equilibrium on any position can be used to stabilize and generate the stable Lissajous orbit.

The controller proposed previously is applied to generate stable Lissajous orbits. Stable Lissajous and Lyapunov orbits for nonresonance are shown in Fig. 3, and resonant periodic orbits (1:2, 1:3, 1:4, 2:3, respectively) are shown in Fig. 4.

Initially, the feedback gains are chosen with $G_1 = 20$, $G_2 = 10$, and $G_3 = 10$, then the gains will be optimized to obtain the same poles.

For position feedback, the gains via optimization are

$$G_1 = 13.0147, \qquad G_2 = 13.0147, \qquad G_3 = 12.0063$$

and the optimization index is improved from 600 to 483 (the improving percentage is 80.5 %).

For full feedback, the gains via optimization are

$$G_1 = 11.3253, \qquad G_2 = 11.3254$$
$$G_3 = 14.4533, \qquad \Delta = -0.5956$$

and the optimization index is improved from 2.0058×10^3 to 1.5587×10^3 (the improving percentage is 77.71%).

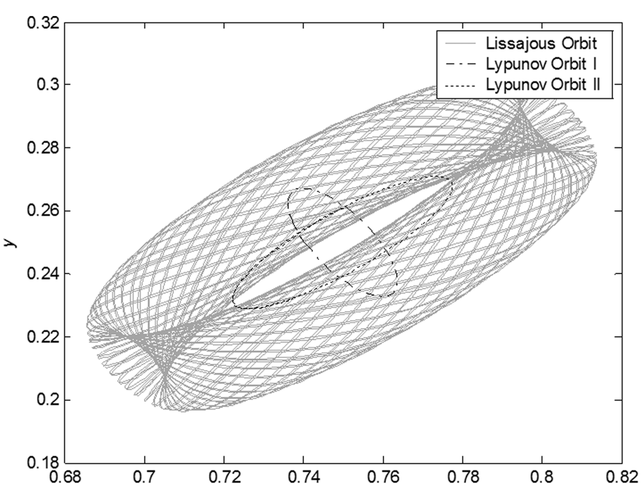

Fig. 3 Stable Lissajous orbit and Lyapunov orbits near the sail equilibrium ($G_1 = 20, G_2 = 10, G_3 = 10$).

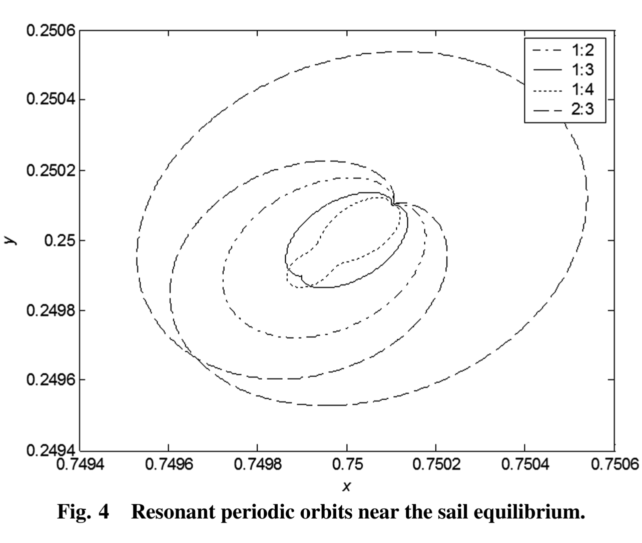

Fig. 4 Resonant periodic orbits near the sail equilibrium.

C. Allocation Law of the Controller

For the 2-DOF solar sail, the acceleration required by the controller is generated by orientating the attitude angle α and changing the sail lightness number β. To investigate the effectiveness of the preceding controller, the values of α and β should be estimated to check whether or not the controller can be implemented in mechanism.

The attitude angle α is defined as the separation angle from $\boldsymbol{r}_1$ to $\boldsymbol{n}$. Then α can be obtained as follows:

$$\begin{cases} \cos\alpha = (\boldsymbol{r}_1 \cdot \boldsymbol{n})/(\|\boldsymbol{r}_1\| \cdot \|\boldsymbol{n}\|) \\ \sin\alpha = (\boldsymbol{r}_1 \times \boldsymbol{n})/(\|\boldsymbol{r}_1\| \cdot \|\boldsymbol{n}\|) \end{cases} \tag{55}$$

From the definition of α, $\boldsymbol{n}$ can be expressed as

$$\boldsymbol{n} = \begin{bmatrix} \cos(\theta + \alpha) & \sin(\theta + \alpha) \end{bmatrix}^T \tag{56}$$

where θ is the separation angle from $[1 \quad 0]^T$ to $\boldsymbol{r}_1$.

Let $\boldsymbol{\Theta} = [\alpha \quad \beta]^T$, then the first-order estimated values of $\boldsymbol{\Theta}$ are formulated as

$$\delta\boldsymbol{\Theta} = \left[\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\Theta}}\right]^{-1}\Bigg|_{z_0} \cdot \boldsymbol{T}_C \tag{57}$$

where

$$\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\Theta}} = \frac{1-\mu}{r_1^2}\begin{bmatrix} -\beta(\sin 2\alpha \cdot n_x + \cos^2\alpha \cdot n_y) & \cos^2\alpha \cdot n_x \\ -\beta(\sin 2\alpha \cdot n_y - \cos^2\alpha \cdot n_x) & \cos^2\alpha \cdot n_y \end{bmatrix}$$

And the changing speed of $\boldsymbol{\Theta}$ can be estimated from

$$\delta\dot{\boldsymbol{\Theta}} = \left[\frac{\partial \boldsymbol{a}}{\partial \boldsymbol{\Theta}}\right]^{-1}\Bigg|_{z_0} \cdot (\boldsymbol{T} \cdot \delta\dot{\boldsymbol{r}} + \boldsymbol{K} \cdot \delta\ddot{\boldsymbol{r}}) \tag{58}$$

The time histories of α and β are shown in Figs 5 and 6, and the speeds at which α and β are changing are shown in Figs. 7 and 8 (the initial parameters of stable Lissajous orbit are the same as that in Fig. 3). The range of α changing is $-3.5^0 \sim 3.5^0$, the range of β changing is just $0.4655 \sim 0.5469$; the speed of α is only in the order of $1.5 \times 10^{-7^0}$ s^{-1}, and the speed of β is only in the order of 1.5×10^{-9} s^{-1}. So the controller can be implemented absolutely in mechanism. β is the sail lightness number, which is the nondimensional factor, and so βs unit should be nondimensional, not in radians or degrees. Similarly, the unit of $\dot{\beta}$ is just s^{-1}.

VIII. Conclusions

A structure-preserving controller has been developed to stabilize the 2-DOF hyperbolic system, and it has also been applied to the

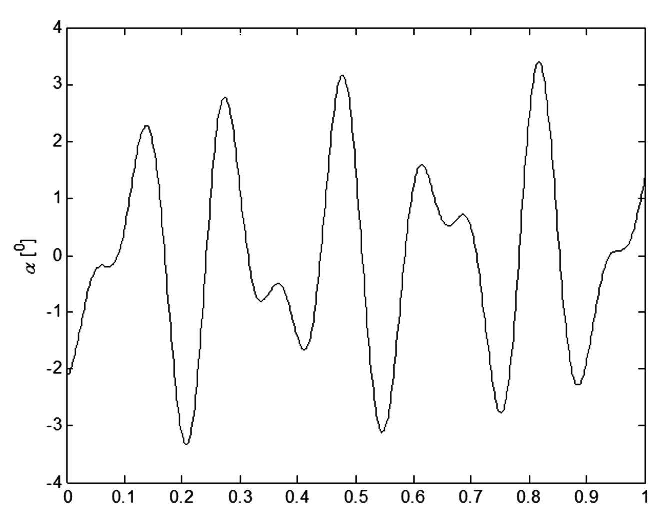

Fig. 5 Time history of α for a stable Lissajous orbit ($G_1 = 20, G_2 = 10, G_3 = 10$).

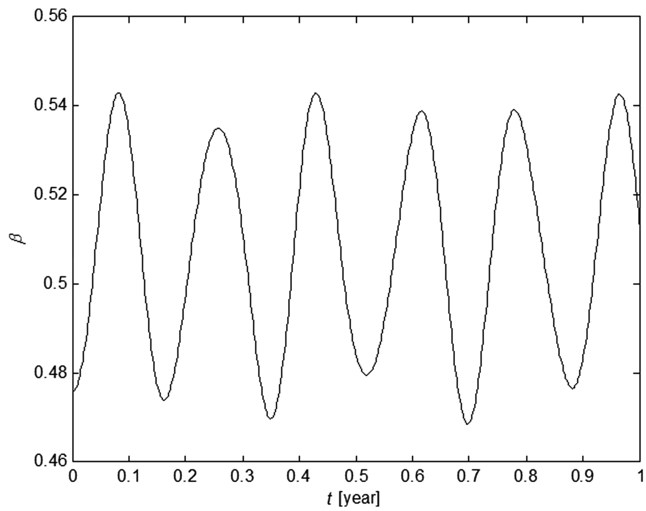

Fig. 6 Time history of β for a stable Lissajous orbit ($G_1 = 20, G_2 = 10, G_3 = 10$).

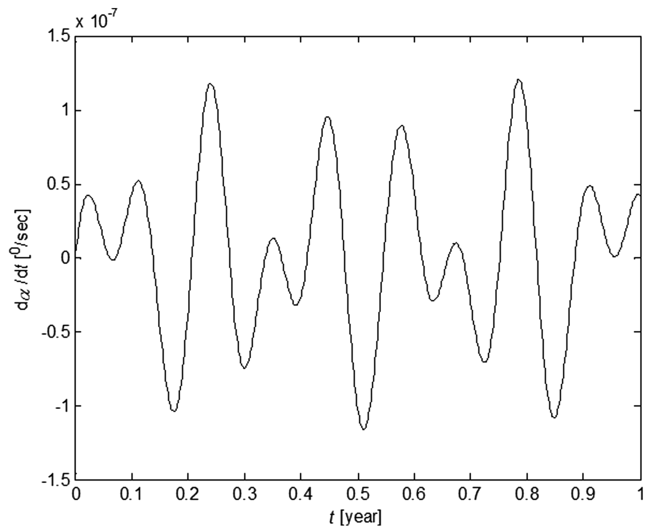

Fig. 7 Time history of αs speed for a stable Lissajous orbit ($G_1 = 20, G_2 = 10, G_3 = 10$).

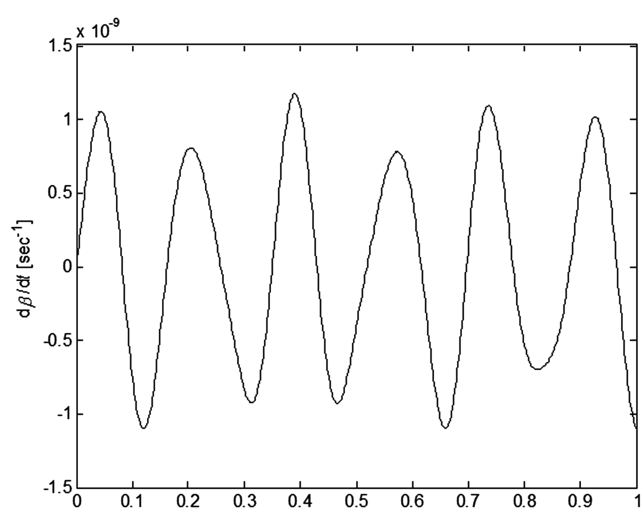

Fig. 8 Time history of βs speed for a stable Lissajous orbit ($G_1 = 20, G_2 = 10, G_3 = 10$).

planar solar sail three-body problem. The nonlinear stability of the controller has been analyzed by means of the Morse lemma in the theory of the Hamiltonian system.

The theorem of pole assignment has been presented and proved. A new type of quasi-periodic orbit, which is referred as stable Lissajous orbit in this paper, has been obtained by means of the controller. The stable Lissajous orbit will degenerate to periodic orbit in the cases of resonance (resonant orbit) and suitable initial values (Lyapunov orbit). A general iterative algorithm has been proposed via differential correction to generate the initial values of Lyapunov orbit. The sensitivity for the controller is measured by the Frobenius norm, which can be used as the optimization index for choosing more suitable values of gains. Applying the controller to the planar solar sail three-body problem yielded the stable Lissajous orbit, which is quite different from the classic Lissajous orbit. Furthermore, the controller can stabilize the sail equilibrium on any position. The investigating results concerning the allocation law of the controller has verified that the controller is realizable in mechanism.

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